Banach-Colmez spaces

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Banach-Colmez spaces were introduced by Colmez in [1] (under the name "*Espaces de Banach de Dimension Finie*") almost fifteen years ago to give a new proof the conjecture "weakly admissible implies admissible" in p-adic Hodge theory. The goal of the talk was to show why they are important and ubiquitous.

Let C be the completion of an algebraic closure of $\mathbf{Q}_p$. Let Perf_C be the category of perfectoid spaces over C, and $\mathrm{Perf}_{C,\mathrm{proet}}$ be the big pro-étale site of C (the above category endowed with the pro-étale topology). We will look at presheaves on the category Perf_C with values in the category of $\mathbf{Q}_p$-topological vector spaces, which are sheaves on $\mathrm{Perf}_{C,\mathrm{proet}}$ when viewed simply as presheaves of $\mathbf{Q}_p$-vector spaces. Such a functor $\mathcal{F}$ is called a *Banach sheaf* when $\mathcal{F}(X)$ is a Banach space for all affinoid perfectoid X. Here are two simple examples of Banach sheaves : the constant sheaf $\underline{V}$, for any finite dimensional $\mathbf{Q}_p$-vector space V ; the sheaf $W \otimes_C \mathcal{O}$, for any finite dimensional C-vector space W.

The following definition looks a bit different from Colmez's one, but is actually equivalent.

Definition 0.1. An *effective Banach-Colmez space* is a Banach sheaf $\mathcal{F}'$ which is an extension [1]

$$0 \to \underline{V} \to \mathcal{F}' \to W \otimes_C \mathcal{O} \to 0,$$

V (resp. W) being a finite dimensional $\mathbf{Q}_p$-vector space (resp. a finite dimensional C-vector space). A *Banach-Colmez space* is a Banach sheaf $\mathcal{F}$ which is a quotient

$$0 \to \underline{V'} \to \mathcal{F}' \to \mathcal{F} \to 0,$$

where $\mathcal{F}'$ is an effective Banach-Colmez space and V' a finite dimensional $\mathbf{Q}_p$-vector space. The category of Banach-Colmez spaces will be denoted $\mathcal{BC}$.

To any presentation of a Banach-Colmez space as in the definition, we associate two integers : its *dimension* $\dim_C W$ and its *height* $\dim_{\mathbf{Q}_p} V - \dim_{\mathbf{Q}_p} V'$.

The definition of the category of Banach-Colmez spaces may look a bit strange, but Colmez proved the following difficult theorem ([1]).

Theorem 0.2. *The category $\mathcal{BC}$ is an abelian category. The functor $\mathcal{F} \mapsto \mathcal{F}(C)$ is exact and conservative on $\mathcal{BC}$.*

Moreover, the functions dimension and height do not depend on the presentation and define two additive functions on $\mathcal{BC}$.

1. By definition, a sequence of Banach sheaves is said to be *exact* if it is so as a sequence of sheaves of $\mathbf{Q}_p$-vectors spaces on $\mathrm{Perf}_{C,\mathrm{proet}}$.

At this point it is not clear that there exist interesting examples of Banach-Colmez spaces apart the obvious ones. To construct geometrically such examples, one can use p-divisible groups, as was observed by Fargues ([2]). Let G be a p-divisible group over $\mathcal{O}_C$. Its *universal cover* $\tilde{G}$ is the sheaf which associates to any perfectoid algebra R over C the $\mathbf{Q}_p$-vector space

$$\tilde{G}(R) = \varprojlim_{\times p} \varprojlim_{k} \varinjlim_{n} G[p^n](R^\circ/p^k).$$

This sheaf is representable by a perfectoid space over C. For example if $G = \mathbf{Q}_p/\mathbf{Z}_p$, $\tilde{G} = \underline{\mathbf{Q}_p}$ [2]. In general, one has an exact sequence of pro-étale sheaves

$$0 \to \underline{V(G)} \to \tilde{G} \xrightarrow{\log_G} \mathrm{Lie}(G)[p^{-1}] \otimes \mathcal{O} \to 0, \tag{1}$$

$V(G)$ being the rational Tate module of G. As moreover $\tilde{G}(R)$ is a Banach space for any perfectoid C-algebra R, this exact sequence shows that universal covers of p-divisible groups are examples of effective Banach-Colmez spaces! Actually, one can prove the following result

Theorem 0.3. *Universal covers of p-divisible groups over $\mathcal{O}_C$ are Banach-Colmez spaces and any Banach-Colmez space is the quotient of the universal cover of a p-divisible group by the Banach-Colmez space $\underline{V}$ associated to some finite dimensional $\mathbf{Q}_p$-vector space V.*

This result has two consequences. The first one is the

Corollary 0.4. *Banach-Colmez spaces are diamonds over* $\mathrm{Spa}(C^\flat)$ [3].

The deep results of Fargues [2] and Scholze-Weinstein [4] on p-divisible groups allow to describe universal covers in terms of p-adic Hodge theory. The second consequence of the theorem is thus that one can get many explicit examples of Banach-Colmez spaces by playing with Fontaine rings. Here is an example : for any $\lambda = d/h \in \mathbf{Q}$, $\lambda \geq 0$, the functor $\mathbf{U}_\lambda : R \mapsto B^+_{\mathrm{cris}}(R^\circ)^{\varphi^h = p^d}$ is a Banach-Colmez space. For instance, $\mathbf{U}_1 = \tilde{\mu}_{p^\infty}$ and the exact sequence (1) for $G = \mu_{p^\infty}$ evaluated on C becomes identified with the famous exact sequence

$$0 \to \mathbf{Q}_p.t \to (B^+_{\mathrm{cris}})^{\varphi = p} \xrightarrow{\theta} C \to 0.$$

To completely elucidate the nature and the structure of the category $\mathcal{BC}$, we now turn to the relation with the Fargues-Fontaine curve X (for $E = \mathbf{Q}_p$, $F = C^\flat$). Let

$$\mathrm{Coh}^{0,-}(X) = \{\mathcal{F} \in D(X), H^i(\mathcal{F}) = 0 \text{ for } i \neq -1, 0; H^{-1}(\mathcal{F}) < 0, H^0(\mathcal{F}) \geq 0\},$$

2. This sheaf is representable by the perfectoid space $\mathrm{Spa}(\mathcal{C}^0(\mathbf{Q}_p, C), \mathcal{C}^0(\mathbf{Q}_p, \mathcal{O}_C))$.
3. Here we implicitly identify $\mathrm{Perf}_{C,\mathrm{proet}}$ with $\mathrm{Perf}_{C^\flat,\mathrm{proet}}$, using Scholze's equivalence.

where $D(X)$ is the bounded derived category of the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X, and where for $\mathcal{E} \in \mathrm{Coh}(X)$, the notation $\mathcal{E} \geq 0$ (resp. $\mathcal{E} < 0$) means that all the slopes of $\mathcal{E}$ are non negative (resp. negative). This full subcategory of $D(X)$ is actually an abelian category (this is a consequence of the general theory of *tilting* and *torsion pairs*), and is endowed with a degree function $\deg^{0,-}$ and a rank function $\mathrm{rk}^{0,-}$.

For any perfectoid space S over $C^\flat$, there exists a relative version X_S of the curve (for $S = \mathrm{Spa}(C^\flat)$, this is just the usual Fargues-Fontaine curve X). Although there is *no* morphism of adic spaces $X_S \to S$, one has a morphism of sites τ from $\mathrm{Perf}_{C^\flat,\mathrm{proet}}$ to the big pro-étale site of X. In particular, one can associate to any complex of coherent sheaves $\mathcal{F}$ on X a sheaf $R^j\tau_*\mathcal{F}$ on $\mathrm{Perf}_{C^\flat,\mathrm{proet}}$, for any $j \geq 0$.

Theorem 0.5. *The functor $R^0\tau_*$ induces an equivalence of categories*

$$\mathrm{Coh}^{0,-}(X) \simeq \mathcal{BC}.$$

Under this equivalence the functions $\deg^{0,-}$ *and* $-\mathrm{ht}$ *(resp.* $\mathrm{rk}^{0,-}$ *and* $\dim$*) correspond to each other.*

For example, $R^0\tau_*$ sends $\mathcal{O}_X$ to $\underline{\mathbf{Q}_p}$, $i_{\infty,*}C$ to $\mathcal{O}$, and $\mathcal{O}_X(-1)[1]$ to $\mathcal{O}/\mathbf{Q}_p$. This result gives a precise meaning to the idea that all Banach-Colmez spaces can be obtained by using H^0 and H^1 of coherent sheaves on the Fargues-Fontaine curve. It also shows that the category $\mathcal{BC}$ only depends on $C^\flat$.

Using this result and the corollary 0.4, one can show that automorphism groups of vector bundles on X are diamonds : see [3, Prop. 2.5]. For example, $\mathcal{A}ut(\mathcal{O}_X^n) = \underline{GL_n(\mathbf{Q}_p)}$ (and not the algebraic group GL_n !). This point is important for Fargues's conjecture.

Références

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